PARAMETRIC STUDY ON THE INSTABILITY CHARACTERISTICS OF A VISCOUS JET

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ABSTRACT

In this article, a linear stability analysis is presented for a liquid jet discharging into a stagnant gas medium. Following the usual approach for a linear analysis, a dispersion relation that relates the amplification factor of a disturbance to its wave number is obtained from the equations of motion for incompressible, viscous, axisymmetric flows in cylindrical coordinates. Surface tension, viscosity of the liquid and electrical charge are the parameters of the problem. It is shown that the surface tension stabilizes the flow in the atomization regime, while the electric charges destabilize it. These results are in agreement with the results of the inviscid analysis. On the other hand, a more viscous liquid jet results in a more stable flow.

INTRODUCTION

The analysis is based on the approach outlined by Reitz and Bracco [1] and starts with the equations of motion for incompressible, viscous, axisymmetric flows in cylindrical coordinates. By using small disturbance theory followed by a normal mode analysis, a dispersion relation is obtained for the dimensionless amplification rate as a function of the disturbance wave number, where the surface tension, density stratification, viscosity of the liquid jet and the electrical charge parameter are the parameters of the problem. Numerical solution of the dispersion relation yields the wave numbers and the amplification factors of the interfacial disturbances. The effect of an electrically charged jet is modeled by applying the approach outlined by Turnbull [2], also used in the inviscid analysis. As such, this study complements the inviscid analysis [3] by investigating the effect of viscosity as an additional parameter.

GOVERNING EQUATIONS

The flow geometry of interest is presented in Figure 1.

Equations of motion for incompressible, axisymmetric flows in cylindrical coordinates are written as:

Momentum equation in the radial (r) direction:

\[
\frac{\partial u_r^*}{\partial t} + u_r^* \frac{\partial u_r^*}{\partial r} + u_z^* \frac{\partial u_r^*}{\partial z} = -\frac{1}{\rho} \frac{\partial p^*}{\partial r} + \nu \left( \frac{\partial^2 u_r^*}{\partial r^2} + \frac{1}{r} \frac{\partial u_r^*}{\partial r} + \frac{\partial u_r^*}{\partial z^2} \right) + \phi \left( \frac{\partial u_r^*}{\partial r} + \frac{\partial u_r^*}{\partial z} \right),
\]

(1)

Momentum equation the axial (z) direction:

\[
\frac{\partial u_z^*}{\partial t} + u_r^* \frac{\partial u_z^*}{\partial r} + u_z^* \frac{\partial u_z^*}{\partial z} = -\frac{1}{\rho} \frac{\partial p^*}{\partial z} + \nu \left( \frac{\partial^2 u_z^*}{\partial r^2} + \frac{1}{r} \frac{\partial u_z^*}{\partial r} + \frac{\partial u_z^*}{\partial z^2} \right) + \phi \left( \frac{\partial u_z^*}{\partial r} + \frac{\partial u_z^*}{\partial z} \right),
\]

(2)

Continuity equation:

\[
\frac{\partial u_r^*}{\partial r} + \frac{u_r^*}{r} + \frac{\partial u_z^*}{\partial z} = 0.
\]

(3)

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The flow is split into steady mean and unsteady perturbation components:

\[ u_1^* = U_1^* \left( r^*, z^* \right) + \dot{u}_r^* \left( r^*, z^*, t^* \right) \]
\[ u_2^* = U_2^* \left( r^*, z^* \right) + \dot{u}_z^* \left( r^*, z^*, t^* \right) \]
\[ p^* = P_0^* \left( r^*, z^* \right) + \dot{p} \left( r^*, z^*, t^* \right) \]  \hspace{1cm} (4)

At this stage, the following assumptions are made:

- Perturbation components are small compared to mean flow counterparts. Therefore, quadratic terms can be neglected.
- The mean flow satisfies the equations of motion.
- The flow is assumed to be parallel, i.e. \( u_1^* = U_1^* \) and \( U_1^* = 0 \).

When the terms given in equation (4) are substituted into equations (1)-(3), a set of linear stability equations is obtained:

\[
\frac{\partial \dot{u}_r}{\partial t} + U_2^* \frac{\partial \dot{u}_r}{\partial z} = \frac{1}{\rho} \frac{\partial \ddot{p}}{\partial r} + \nu \left( \frac{\partial^2 \dot{u}_r}{\partial r^2} + \frac{1}{r} \frac{\partial \dot{u}_r}{\partial r} + \frac{\partial^2 \dot{u}_r}{\partial z^2} \right) \]  \hspace{1cm} (5)

\[
\frac{\partial \dot{u}_z}{\partial t} + U_2^* \frac{\partial \dot{u}_z}{\partial z} = \frac{1}{\rho} \frac{\partial \ddot{p}}{\partial z} + \nu \left( \frac{\partial^2 \dot{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \dot{u}_z}{\partial r} + \frac{\partial^2 \dot{u}_z}{\partial z^2} \right). \]  \hspace{1cm} (6)

\[
\frac{\partial \dot{u}_r}{\partial r} + \frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial z} = 0. \]  \hspace{1cm} (7)

In the following, subscript 1 indicates the liquid, while 2 indicates the gas phases, respectively. In the current approach the coordinate moves with the velocity of the interface, therefore \( U_1^* (r^*) = 0 \), \( U_2^* (r^*) = U_o^* \) and the viscosity of the gas is neglected, \( \nu_2^* = 0 \).

Interface conditions:

- The fluid particles at the liquid-gas interface remain there:
  \[ \dot{u}_{1r} = \frac{\partial \eta^*_r}{\partial t}, \]  \hspace{1cm} (8)
  \[ \dot{u}_{2r} = \frac{\partial \eta^*_r}{\partial t} + U_o^* \frac{\partial \eta^*_r}{\partial z}. \]  \hspace{1cm} (9)

- The pressure difference between the liquid and gas phases is balanced by viscous stresses, surface tension and the stresses due to electrical charges:
  \[ -\ddot{p}_1^* + 2\mu_1^* \frac{\partial \dot{u}_r^*}{\partial z} = -\ddot{p}_2^* + \sigma_e \frac{\partial^2 \eta^*_r}{\partial z^2} + \frac{\sigma_o^2 \eta^*_r}{\partial z^2} + \frac{\sigma_o^2 \eta^*_r}{\partial z^2} - \frac{\alpha_e a^*}{K_o (\alpha a^*) - 1}. \]  \hspace{1cm} (10)
• The shear stress is continuous at the liquid-gas interface. This principle can be expressed as follows:

\[
\mu \left( \frac{\partial \hat{u}_r^*}{\partial r} + \frac{\partial \hat{u}_z^*}{\partial z} \right) - \frac{\sigma^2_{\theta}}{v_0} \bar{\alpha} \eta^* = 0.
\]  
(11)

From this point on, the analyses for the liquid and gas phases should be carried out separately.

Solution for the liquid phase

The set of linear equations for the liquid phase are as follows:

\[
\frac{\partial \hat{u}_r^*}{\partial t} = -\frac{1}{\rho_1} \frac{\partial \hat{p}^*}{\partial r} + v_1 \left( \frac{\partial^2 \hat{u}_r^*}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{u}_r^*}{\partial r} + \frac{\partial^2 \hat{u}_z^*}{\partial z^2} \right),
\]  
(12)

\[
\frac{\partial \hat{u}_z^*}{\partial t} = -\frac{1}{\rho_1} \frac{\partial \hat{p}^*}{\partial z} + v_1 \left( \frac{\partial^2 \hat{u}_z^*}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{u}_z^*}{\partial r} + \frac{\partial^2 \hat{u}_z^*}{\partial z^2} \right),
\]  
(13)

\[
\frac{\partial \hat{u}_r^*}{\partial r} + \frac{\hat{u}_r^*}{r} + \frac{\hat{u}_z^*}{\partial z^2} = 0.
\]  
(14)

At this stage, a stream function and a velocity potential are defined as follows:

\[
\Phi_1^* \left( \hat{r}^*, \hat{z}^*, t^* \right) = \phi_1^* \left( \hat{r}^* \right) e^{i \hat{z}^* - c^* \hat{r}^*},
\]  
(15)

\[
\Psi_1^* \left( \hat{r}^*, \hat{z}^*, t^* \right) = \psi_1^* \left( \hat{r}^* \right) e^{i \hat{z}^* - c^* \hat{r}^*}.
\]  
(16)

According to Helmholtz decomposition, each perturbation velocity component is separated into a rotational and an irrotational component. Using the velocity potential and the stream function, these can be expressed as follows:

\[
\hat{u}_r^* = \frac{\partial \Phi_1^*}{\partial r} + \frac{1}{r} \frac{\partial \Psi_1^*}{\partial z},
\]  
(17)

\[
\hat{u}_z^* = \frac{\partial \Phi_1^*}{\partial z} - \frac{1}{r} \frac{\partial \Psi_1^*}{\partial r}.
\]  
(18)

Solution of the velocity potential:

When the pressure term is eliminated between equations (12) and (13), a single equation is obtained, which can be expressed in terms of the velocity potential as follows:

\[
\phi_1^* + \frac{\hat{\phi}_1^*}{r} - \alpha^* \phi_1^* = 0.
\]  
(19)

The general solution of this equation is as follows:

\[
\phi_1^* = c_1 I_0 \left( \alpha^* \hat{r} \right) + c_2 K_0 \left( \alpha^* \hat{r} \right).
\]  
(20)

However, \( \phi_1^* \) must be finite as \( \hat{r}^* \rightarrow 0 \), implying \( c_2 = 0 \). Therefore:

\[
\phi_1^* = c_1 I_0 \left( \alpha^* \hat{r} \right).
\]  
(21)

Solution of the stream function:

When the pressure term is eliminated between equations (12) and (13), a single equation is obtained, which can be expressed in terms of the stream function as follows:

\[
\psi_1^* - \frac{\hat{\psi}_1^*}{r^*} - \left( \alpha^* \omega^* + \frac{\omega^*}{v_1^*} \right) \psi_1^* = 0.
\]  
(22)

Here, the circular frequency is defined as \( \omega^* = -i \alpha^* c^* \). The general solution of equation (22) is:

\[
\psi_1^* = c_3 \hat{r}^* I_1 \left( \alpha^* \hat{r} \right) + c_2 \hat{r}^* K_1 \left( \alpha^* \hat{r} \right).
\]  
(23)

In the above expression, \( \alpha^* = \alpha^* + \omega^*/v_1^* \).
However, $\psi_1^*$ must be finite as $r^* \to 0$, requiring $c_2 = 0$. Therefore the solution attains the form:

$$\psi_1^* = c_2 r^* h_1(\alpha^* r^*)^2.$$  \hfill (24)

When equations (21) and (24) are used in equation (8) together with $\eta_1^* = \eta_0^* e^{i \alpha^* z^* + \omega^* t^*}$ the following expression is obtained at $r^* = \hat{a}$:

$$c_1 \alpha^* h_1(\alpha^* \alpha^*) + i \alpha^* c_2 h_1(\alpha^* \alpha^*) = \omega^* \eta_0^*.$$  \hfill (25)

If equations (21) and (24) are substituted into equation (11) the following expressions are obtained:

$$c_1 = \frac{\alpha^* \eta_0^* - \alpha^* \eta_0^* - \frac{1}{\epsilon_o} - \frac{\epsilon_o}{\mu_1}}{\epsilon_o^2 h_1(\alpha^* \alpha^*)},$$

$$c_2 = \frac{\alpha^* \eta_0^* - \frac{1}{\epsilon_o}}{(\alpha^* - \alpha^*) h_1(\alpha^* \alpha^*)},$$  \hfill (26)

As a result of this the velocity potential and the streamfunction become:

$$\phi_1^* = \frac{\alpha^* \eta_0^*}{\alpha^* h_1(\alpha^* \alpha^*)} - \frac{1}{\epsilon_o} + \frac{\epsilon_o}{\mu_1} f_1(\alpha^* r^*) e^{i \alpha^* z^* + \omega^* t^*},$$

$$\psi_1^* = \frac{\alpha^* \eta_0^*}{\alpha^* - \alpha^*} \frac{1}{\epsilon_o} + \frac{\epsilon_o}{\mu_1} f_1(\alpha^* r^*) e^{i \alpha^* z^* + \omega^* t^*}.$$  \hfill (28)

The pressure in the liquid phase can be found by using only the inviscid terms of equation (12):

$$\frac{\partial^2 h}{\partial t^2} = \frac{\rho_1}{\partial r^2}.$$  \hfill (30)

When equations (17) and (28) are used in the above expression, the pressure distribution in the liquid phase is obtained as follows:

$$\hat{p}_1^* = \frac{\alpha^*}{\alpha^* h_1(\alpha^* \alpha^*)} - \frac{1}{\epsilon_o} + \frac{\epsilon_o}{\mu_1} f_1(\alpha^* r^*) e^{i \alpha^* z^* + \omega^* t^*}.$$  \hfill (31)

**Solution for the gas phase**

The linear set of equations for the gas phase is:

$$\frac{\partial^2 U_2}{\partial t^2} + U_2 \frac{\partial^2 U_2}{\partial z^2} = \frac{1}{\rho_2} \frac{\partial \hat{p}_2^*}{\partial r^*},$$  \hfill (32)

$$\frac{\partial^2 U_2}{\partial t^2} + \frac{\partial U_2}{\partial r^*} + U_2 \frac{\partial^2 U_2}{\partial z^2} = \frac{1}{\rho_2} \frac{\partial \hat{p}_2^*}{\partial r^*},$$  \hfill (33)

$$\frac{\partial^2 U_2}{\partial r^*} + \frac{\partial U_2}{\partial z^2} = 0.$$  \hfill (34)

A streamfunction is defined for the gas phase:

$$\psi_2^* = \left[ U_2^* - i \frac{\alpha^*}{\alpha^*} \right] f(r^*).$$  \hfill (35)

When the pressure term is eliminated between equations (32) and (33), and when $U_2^*(r^*) = U_0^*$ is used the following expression is obtained:
If the velocity components in this equation are expressed in terms of the streamfunction given in equation (35), the following ordinary differential equation is obtained:

\[ f^* - \frac{f^*}{r} - \alpha^2 f = 0. \]  

(37)

The general solution of this equation in terms of Bessel functions is as follows:

\[ f(r^*) = c_1 r^r I_1(\alpha^* r^*) + c_2 r^r K_1(\alpha^* r^*). \]  

(38)

However, \( f(r^*) \) should remain finite as \( r^* \to \infty \). Therefore \( c_1 = 0 \) and the solution becomes:

\[ f(r^*) = c_2 r^r K_1(\alpha^* r^*). \]  

(39)

In order to find the integration constant \( c_2 \), equation (9) and the expression \( \eta^* = \eta_0^* e^{i(\alpha^* z + \omega^* t^*)} \) are utilized. Accordingly \( c_2 = 1/K_1(\alpha^* a^*) \) is obtained. Therefore:

\[ f(r^*) = \frac{r^r K_1(\alpha^* r^*)}{K_1(\alpha^* a^*)}. \]  

(40)

If the terms in equations (32) are expressed in terms of the streamfunction defined in equations (35) and (40) the pressure field in the gas phase is obtained as follows:

\[ \hat{P}_2^* = -p_2 \left( \frac{\alpha^*}{\alpha} \right)^2 \frac{\eta^* K_0(\alpha^* r^*)}{K_1(\alpha^* a^*)}. \]  

(41)

In the last step, if equations (31) and (41) are substituted into equation (10), the following dispersion relation is obtained after rearrangements:

\[ \omega^2 + 2\nu_1 \alpha^2 \omega^* \left[ \frac{I_1(\alpha^*)}{I_0(\alpha^*)} - \frac{2\alpha^*}{\alpha^2 + \alpha^2 I_0(\alpha^*)} \right] \left[ \frac{\rho_0}{\varepsilon_0 \nu_1} \frac{1}{\alpha^2 + \alpha^2 I_0(\alpha^*)} \right] = \frac{\sigma^*}{\rho_1 \alpha^2} \left( \frac{\alpha^*}{\alpha} \right)^2 \frac{\alpha^2 - \alpha^2 I_0(\alpha^*)}{\alpha^2 + \alpha^2 I_0(\alpha^*)} \frac{K_0(\alpha^* a^*)}{K_1(\alpha^* a^*)} \]  

(42)

Non-dimensionalization of the above equation is accomplished by multiplying all terms with the group \( \rho_1 \alpha^3 / \sigma^* \). Accordingly, the non-dimensional dispersion relation is obtained as follows:

\[ \beta^2 + 2\beta \omega_0^2 \left[ \frac{I_1(\alpha)}{I_0(\alpha)} - \frac{2\alpha}{\alpha^2 + \alpha^2 I_0(\alpha)} \frac{I_1(\alpha)}{I_1(\alpha)} \right] - \frac{1}{2} E_1 \frac{1}{\alpha^2 + \alpha^2} \]  

\[ = \alpha(1 - \alpha^2) \frac{\alpha^2 - \alpha^2}{\alpha^2 + \alpha^2 I_0(\alpha)} + \frac{\beta^2}{\alpha^2 + \alpha^2 I_0(\alpha)} \frac{K_0(\alpha^2) - 1}{\alpha^2 + \alpha^2 I_0(\alpha)} \]  

(43)
The non-dimensional groups appearing in the dispersion relation are defined as follows:

\[ \beta = \omega * \left( \frac{\rho_1^* a^*}{\sigma^*} \right)^3; \text{non-dimensional amplification rate,} \]  
(44)

\[ Oh = \frac{\mu_1^*}{\sqrt{\rho_1^* \sigma^* a}}; \text{Ohnesorge number,} \]  
(45)

\[ We = \frac{\rho_2^* U_0^2 a^*}{\sigma^*}; \text{Weber number,} \]  
(46)

\[ E_1 = \frac{\sigma_o^* a^*}{\varepsilon_o \mu_1^*}; \]  
(47)

\[ E_1 = \frac{\sigma_o^* a^*}{\varepsilon_o \sigma^*}. \]  
(48)

**SOLUTION METHOD, RESULTS AND DISCUSSION**

For the solution of equation (43) a computer code using the Fortran 77 programming language is developed. The solution method is based on finding \( \beta \), which is the root of the equation, as a function of the parameters listed in equations (44)-(48). In order to achieve this, the simplex method, which uses only the function values in order to find its global minima and maxima, is employed [4]. The values of the Bessel functions are computed by utilizing the subroutines given by the same reference.

The parametric study is handled by using the dimensional forms of the physical variables. The reason for this is that the Weber and Ohnesorge numbers are not independent of each other. For example if the surface tension is varied, both the Weber and the Ohnesorge numbers are affected.

The baseline configuration for the parametric study is a water-air system for which the parameters are as follows:

\[ \rho_1^* = 999 \text{kg/m}^3; \text{density of water,} \]  

\[ \rho_2^* = 1.225 \text{kg/m}^3; \text{density of air,} \]  

\[ \sigma^* = 0.073 \text{N/m}; \text{surface tension of water,} \]  

\[ \mu_1^* = 1 \times 10^{-3} \text{kg/m.s}; \text{viscosity of water,} \]  

\[ a^* = 250 \mu \text{m}; \text{radius of the jet,} \]  

\[ \varepsilon_o^* = 8.854 \times 10^{-12} \text{C}^2/\text{N.m}^2; \text{dielectric constant,} \]  

\[ U_0^* = 70 \text{m/s}; \text{jet exit velocity.} \]  

Figure 2 shows the effect of surface tension on the instability characteristics. As can be seen, as the surface tension increases, the flow is stabilized. Not only the amplification rate levels decrease, but also the range of unstable wavenumbers gets narrower.

In Figure 3, the effect of liquid viscosity on the instability characteristics is depicted. Increasing liquid viscosity renders the flow more stable. However, when compared with the surface tension, the stabilizing effect of viscosity is much weaker. When figure 2 is observed once more, it can be seen that increasing the surface tension from 0.05N/m to 0.1N/m reduces the dimensionless amplification rate by two-thirds. In order to achieve the same effect with liquid viscosity, one needs to increase the viscosity by tenfold.

Figure 4 illustrates the effect of electric charges at the surface of the jet on the instability characteristics. As the electric charge density is increased, the flow is destabilized.
Figure 2: Effect of surface tension on the amplification rates.

Figure 3: Effect of liquid viscosity on the amplification rates.
CONCLUSIONS

A parametric study is performed to bring out the effects of surface tension, viscosity and electrical charges on the instability of a liquid jet flow. It has been shown that surface tension and higher liquid viscosity stabilize the flow. An electrically charged liquid enhances the jet flow instability. These inferences can be put into practical use in internal combustion engines and turbomachinery. As smaller fuel droplets are desired for more efficient combustion, a more unstable jet flow is preferred. For example the liquid can be electrically charged before being discharged from the nozzle. Also, the surface tension can be reduced by adding surface active materials to the fuel so that the resulting flow is more unstable.

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References